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ON NON-ISOTHERMAL ELASTIC-PLASTIC AND ELASTIC-VISCOPLASTIC DEFORMATIONS

R. DE BOER
Universität Essen, W. Germany†

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Abstract—Starting with the equations of balance of energy and the Clausius-Duhem inequality the non-isothermal behavior of elastic-plastic materials without and with viscous properties is described. All quantities in the equations of balance of energy and in the Clausius-Duhem inequality are expanded in series. This procedure leads to the development of restrictions and stress-strain-relations, which contain as special cases the constitutive equations of classical plasticity and viscoplasticity.

1. INTRODUCTION

The thermodynamical behavior of continua is described by the conservation equations and the constitutive equations in the framework of classical continuum mechanics. Invariance requirements and the Clausius-Duhem inequality act as control quantities. Whereas the equations of balance and certain invariance conditions may be considered as valid, the form of the Clausius-Duham inequality is criticized principally (see, e.g. Perzyna [11]). A motivation for the use of the Clausius-Duhem inequality to derive restrictions on the constitutive equations is surely given by virtue of getting important results for constitutive equations of certain material behavior, as e.g. in the field of fluid mechanics and in elasticity theory (see Fung[2]). In this paper we also assume the validity of the Clausius–Duhem inequality to derive restrictions on the constitutive equations for elastic-plastic materials without and with viscous properties.

Important progress in the development of constitutive equations has already been derived for elastic-plastic materials (Green-Naghdi[3], Lee[5], Naghdi-Trapp[8]) and for a certain class of elastic-viscoplastic materials (Perzyna[11]) in connection with a general nonlinear theory. The aim of this paper is to extend the development of constitutive equations and restrictions for elastic-plastic behavior of materials with and without viscous properties, where we also include such a viscoplastic behavior as creep. For a special class of viscoplastic deformations we consider unloading which leads to a different approach of the derivation of constitutive equations for viscoplastic behavior. We regard the theory of elastic-plastic materials with and without viscous properties as a "rate-type" theory. For our purposes therefore it is necessary to develop the change of energy and entropy in a Taylor series expansion. Using the series expansion, we obtain various cases of the equation of balance of energy and of the Clausius-Duhem inequality. We assume that all cases have to be satisfied by the constitutive equations, describing the elastic-plastic and elastic-viscoplastic behavior of the material.

Before we start with the description of non-isothermal deformations of elastic-plastic material, the content of the paper should be indicated. Basic equations are given in Section 2, in which we also consider the development of the change of energy and entropy in a Taylor series expansion. The full system of constitutive equations describing the behavior of an elastic-plastic material without and with viscous properties is given in Sections 3 and 4. Section 3 contains the description of inviscid elastic-plastic materials. One can show, that in the loading criteria a basic inequality is involved which allows the derivation of an explicit form of the constitutive equation for the plastic strain rates. In special cases, this form of the constitutive equation leads to the normality of the plastic strain rates in stress space. Section 4 describes elastic-plastic materials with viscous properties.

2. BASIC EQUATIONS

Let us consider a body $\mathscr{B}$ with material particles X and identify the material particles X with its position $\mathbf{X}$ in a reference configuration $\mathscr{R}$. All following quantities that shall be

†Presend address: Stiftung Volkswagenwerk scholar, University of California, Berkeley, California, U.S.A.

introduced are referred to this reference configuration $\mathscr{R}$. The body may deform and conduct heat. The motion of the body is described by a smooth vector function

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \tag{2.1}$$

where $\mathbf{x}$ is the spatial position of $\mathscr{B}$ at time t.

The deformation graditent $\mathbf{F}$ is determined by

$$\mathbf{F} = \nabla\boldsymbol{\chi}(\mathbf{X}, t), \tag{2.2}$$

where the gradient ∇ is computed with respect to the material coordinates X, keeping t fixed. It is assumed that

$$\mathscr{T} = \det \mathbf{F} > 0. \tag{2.3}$$

The condition for the balance of linear momentum inside the body in local form is

$$\operatorname{div}(\mathbf{FS}) + \rho\mathbf{b} = \rho\ddot{\mathbf{x}} \tag{2.4a}$$

with the symmetric Piola–Kirchhoff stress tensor $\mathbf{S}(\mathbf{X}, t)$ and the body forces per unit mass $\mathbf{b}(\mathbf{X}, t)$. The mass density in the reference configuration $\mathscr{R}$ is denoted by ρ. The operator div refers to the material coordinates $\mathbf{X}$. The superposed dot denotes the time derivative with respect to the time t keeping $\mathbf{X}$ fixed. The stresses also must satisfy the boundary conditions

$$\mathbf{F}\,\mathbf{Sn} = \mathbf{p}, \tag{2.4b}$$

where $\mathbf{n}$ is the outward unit normal to the surface in the reference configuration and $\mathbf{p}$ the surface force vector, measured per unit area in the reference configuration. The theory of elastic-inelastic bodies may be a theory of increments (Green-Naghdi [3], Koiter [4]). Therefore, the increments of stresses and forces also must fulfill the eqns (2.4). We introduce the balance of energy (the first law of thermodynamics) in local form:

$$\Delta\epsilon = \Delta W + \Delta q. \tag{2.5}$$

The first law of thermodynamics states that the change of the internal energy $\Delta\epsilon = \Delta\epsilon(\mathbf{X}, t)$ of a thermodynamic system in the time interval $t_2 - t_1$, is equal to the sum of the mechanical work performed, $\Delta w = \Delta w(\mathbf{X}, t)$ and of the heat supplied in the time interval $t_2 - t_1$, $\Delta q = \Delta q(\mathbf{X}, t)$. $\Delta\epsilon$ is the difference of the internal energy at time t_2 and t_1 ($\Delta\epsilon = \rho\epsilon(\mathbf{X}, t_2) - \rho\epsilon(\mathbf{X}, t_1)$. where ϵ is the internal energy per unit mass. The Clausius–Duhem inequality (in further considerations called C-D inequality) may be written as:

$$\Delta\eta \geq \frac{\Delta q}{\theta}, \tag{2.6}$$

where $\Delta\eta$ is the difference of entropy at time t_2 and at $t_1(\Delta\eta = \rho\eta(\mathbf{X}, t_1) - \rho\eta(\mathbf{X}, t_2))$ and where η denotes the entropy per unit mass. The quantity $(\Delta q/\theta) = ([\Delta q(\mathbf{X}, t)]/[\theta(\mathbf{X}, t])$ is the entropy flux and $\theta(\mathbf{X}, t)$ is the local absolute temperature which is assumed to positive. All introduced equations must be satisfied for every particle in $\mathscr{B}$. Further, we shall assume that all quantities satisfy the invariance conditions under superposed rigid body motions. For further considerations we need the explicit form of the mechanical work and of the heat. The mechanical work is given by

$$\Delta W = \int_{t_1}^{t_2} tr(\mathbf{S}\dot{\mathbf{E}})\mathrm{d}t, \tag{2.7}$$

where $\mathbf{E}(\mathbf{X}, t)$ denotes the Langrangian strain tensor

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}), \tag{2.8}$$

$\mathbf{F}^T$ is the transpose of $\mathbf{F}$ and $\mathbf{1}$ the unit tensor. The heat is expressed by

$$\Delta q = -\int_{t_1}^{t_2} (\text{div}\,\dot{\mathbf{q}} - \rho\dot{r})\text{d}t \tag{2.9}$$

and the entropy flux

$$\frac{\Delta q}{\theta} = -\int_{t_1}^{t_2} \left[\text{div}\left(\frac{\dot{\mathbf{q}}}{\theta}\right) - \rho\frac{\dot{r}}{\theta}\right]\text{d}t. \tag{2.10}$$

The heat flux vector $\dot{\mathbf{q}}(\mathbf{X}, t)$, referred to the reference configuration, is denoted per unit mass and unit time and $\dot{r}(\mathbf{X}, t)$ is the heat supply per unit mass and unit time.†

With (2.7) (2.9) and (2.10) the equation of balance of energy and the C-D inequality ((2.5) and (2.6)) can be written as

$$\rho\epsilon(\mathbf{X}, t_2) - \rho\epsilon(\mathbf{X}, t_1) = \int_{t_1}^{t_2} [tr(\mathbf{S}\dot{\mathbf{E}}) - \text{div}\,\dot{\mathbf{q}} + \rho\dot{r}]\,\text{d}t, \tag{2.11}$$

$$\rho\eta(\mathbf{X}, t_2) - \rho\eta(\mathbf{X}, t_1) \geqq -\int_{t_1}^{t_2} \left[\text{div}\left(\frac{\dot{\mathbf{q}}}{\theta}\right) - \rho\frac{\dot{r}}{\theta}\right]\text{d}t. \tag{2.12}$$

In this form (2.11) and (2.12) may be useful if one regards closed cycles in the stress-or strain space. For our purpose, however, we express all quantities in (2.11) and (2.12) by a Taylor series expansion. We start from the reference configuration at time t_0. For $t = t_0$ the quantities $\mathbf{S} = \mathbf{S} = 0$, $\mathbf{E} = \mathbf{E}_0$, $\theta = \theta_0$, $\dot{\mathbf{q}} = \dot{\mathbf{q}}_0 = 0$, and $\dot{r} = \dot{r}_0 = 0$ may be given. An external agency takes the body to the state at time t_1 and to the state at time t_2 near the state at time t_1. Then, by a Taylor series expansion we obtain, assuming that all quantities are continously differentiable with respect to the time:

$$\rho\epsilon(\mathbf{X}, t_2) = \rho\epsilon(\mathbf{X}, t_1) + \rho\sum_{n=1}^{\infty} D^n\epsilon(\mathbf{X}, t_1)\cdot(t_2 - t_1)^n\frac{1}{n!}, \tag{2.13}$$

$$\rho\eta(\mathbf{X}, t_2) = \rho\eta(\mathbf{X}, t_1) + \rho\sum_{n=1}^{\infty} D^n\eta(\mathbf{X}, t_1)\cdot(t_2 - t_1)^n\frac{1}{n!}. \tag{2.14}$$

D^n‡ stands for the n-th material time derivative. Also

$$tr(\mathbf{S}\dot{\mathbf{E}}) = tr(\mathbf{S}\dot{\mathbf{E}})\rfloor_{t_1} + \sum_{n=1}^{\infty} D^n tr(\mathbf{S}\dot{\mathbf{E}})\rfloor_{t_1}\cdot(t_2 - t_1)^n\frac{1}{n!}, \tag{2.15}$$

$$\text{div}\,\dot{\mathbf{q}} = \text{div}\,\dot{\mathbf{q}}\rfloor_{t_1} + \sum_{n=1}^{\infty} D^n\,\text{div}\,\dot{\mathbf{q}}\rfloor_{t_1}\cdot(t_2 - t_1)^n\frac{1}{n!}, \tag{2.16}$$

$$\dot{r} = \dot{r}\rfloor_{t_1} + \sum_{n=1}^{\infty} D^n\dot{r}\rfloor_{t_1}\cdot(t_2 - t_1)^n\frac{1}{n!}, \tag{2.17}$$

and

$$\text{div}\left(\frac{\dot{\mathbf{q}}}{\theta}\right) = \text{div}\left(\frac{\dot{\mathbf{q}}}{\theta}\right)\bigg]_{t_1} + \sum_{n=1}^{\infty} D^n\text{div}\left(\frac{\dot{\mathbf{q}}}{\theta}\right)\bigg]_{t_1}\cdot(t_2 - t_1)^n\frac{1}{n!}, \tag{2.18}$$

$$\frac{\dot{r}}{\theta} = \frac{\dot{r}}{\theta}\bigg]_{t_1} + \sum_{n=1}^{\infty} D^n\left(\frac{\dot{r}}{\theta}\right)\bigg]_{t_1}\cdot(t_2 - t_1)^n\frac{1}{n!}. \tag{2.19}$$

†Usually, the heat flux vector and the heat supply are denoted by **q** or **h**, respectively. It seems to be more convenient in view of the Taylor series expansion to use the introduced notation.

‡Only in the series expansion we use the symbol D^n. Later we replace D^n by dots.

Hence, we obtain the rate of energy and entropy using eqn (2.11) and eqn (2.12) in explicit form†

$$\rho \sum_{n=1}^{\infty} D^n \epsilon (t_2 - t_1)^n \frac{1}{n!} = \sum_{n=1}^{\infty} D^{n-1}\, tr(\mathbf{S}\dot{\mathbf{E}})\cdot(t_2-t_1)^n \frac{1}{n!} - \sum_{n=1}^{\infty} D^{n-1} \operatorname{div} \dot{\mathbf{q}}\cdot(t_2-t_1)^n \frac{1}{n!}$$
$$- \rho \sum_{n=1}^{\infty} D^{n-1}\, \dot{r}\cdot(t_2-t_1)^n \frac{1}{n!}, \tag{2.20}$$

$$\rho \sum_{n=1}^{\infty} D^n \eta (t_2 - t_1)^n \frac{r}{n!} \geq - \sum_{n=1}^{\infty} D^{n-1} \operatorname{div}\left(\frac{\dot{\mathbf{q}}}{\theta}\right)\cdot(t_2-t_1)^n \frac{1}{n!} + \rho \sum_{n=1}^{\infty} D^{n-1}\left(\frac{\dot{r}}{\theta}\right)\cdot(t_2-t_1)^n \frac{r}{n!}. \tag{2.21}$$

We postulate that an elastic-inelastic material has to fulfil all basic equations described in Section 2 as well as the equation of balance of energy and the C-D inequality.

Starting from the equation of balance of energy and the C-D inequality (2.20) and (2.21) we can derive important special forms for various cases:

Case (*a*). Taking the value n equal one we derive from (2.20) and (2.21):

$$\rho\dot{r} - \rho\dot{\epsilon} - \operatorname{div}\dot{\mathbf{q}} + tr(\mathbf{S}\dot{\mathbf{E}}) = 0, \tag{2.22}$$

$$\rho\dot{\eta} + \operatorname{div}\left(\frac{\dot{\mathbf{q}}}{\theta}\right) - \rho\frac{\dot{r}}{\theta} \geq 0. \tag{2.23}$$

For further considerations it is convenient to introduce the Helmholtz free energy function per unit mass:

$$\psi = \epsilon - \eta\theta. \tag{2.24}$$

Then, if we use ϵ from (2.24) in (2.22) we obtain

$$\rho\dot{r} - \rho(\dot{\Psi} + \dot{\eta}\theta + \eta\dot{\theta}) - \operatorname{div}\dot{\mathbf{q}} + tr(\mathbf{S}\dot{\mathbf{E}}) = 0. \tag{2.25}$$

Substituting $\dot{\eta}$ form (2.25) into (2.23) the C-D inequality takes the following form:

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + tr(\mathbf{S}\dot{\mathbf{E}}) - \dot{\mathbf{q}}\frac{1}{\theta}\mathbf{g} \geq 0, \tag{2.26}$$

where $\mathbf{g}$ stands for grad θ. (2.25) and (2.26) are the well known forms of the equation of balance of energy and the C-D-inequality.

Case (*b*). Taking into account eqn (2.22), we get from (2.20) the equation of balance of energy in the following form ($n = 2$):

$$\rho\ddot{r} - \rho(\ddot{\psi} + 2\dot{\eta}\dot{\theta} + \ddot{\eta}\theta + \eta\ddot{\theta}) - \operatorname{div}\ddot{\mathbf{q}} + tr(\mathbf{S}\dot{\mathbf{E}})^{\cdot} = 0. \tag{2.27}$$

In this equation the internal energy function is replaced by the Helmholtz free energy function ψ(see eqn (2.24)). Since the term with $n = 1$ in C-D-inequality (2.21) is assumed to be non-negative (eqn (2.23)), ‡ the entire term up to $n = 2$ in (2.21) is surely non-negative, if we assume

$$\rho\ddot{\eta} + \operatorname{div}\left(\frac{\dot{\mathbf{q}}}{\theta}\right)^{\cdot} - \rho\left(\frac{\dot{r}}{\theta}\right)^{\cdot} \geq 0. \tag{2.28}$$

†We suppress the index t_1 from various expressions.

‡The introduction of this strong restriction is not necessary, as we will see, for the description of viscoplastic materials (see Section 4.2).

With the substitution of $\ddot{\eta}$ from (2.27) we are able to express (2.27) in the following form

$$\rho\ddot{r} - \rho(\ddot{\Psi} + 2\dot{\eta}\dot{\theta} + \eta\ddot{\theta}) + tr(\mathbf{S}\dot{\mathbf{E}})^{\cdot} - \operatorname{div}\ddot{\mathbf{q}} + \theta \operatorname{div}\left(\frac{\dot{\mathbf{q}}}{\theta}\right)^{\cdot} - \theta_{\rho}\left(\frac{\dot{r}}{\theta}\right)^{\cdot} \geq 0. \tag{2.29}$$

In general, for a complete description of elastic-inelastic materials in the framework of non-isothermal deformations, higher terms in the Taylor series expansion may be important. For the development in this paper however, we limit our further considerations by the forms (2.25) and (2.26), (2.27) and (2.29) respectively.

3. ELASTIC-PLASTIC MATERIALS

We consider elastic-plastic materials which may be characterized by a set of constitutive eqns (3.1)–(3.18):

$$\begin{aligned} \psi &= \bar{\Psi}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g}) \\ \eta &= \bar{\eta}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g}) \\ \mathbf{S} &= \bar{\mathbf{S}}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g}) \\ \dot{\mathbf{q}} &= \bar{\dot{\mathbf{q}}}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g}) \end{aligned} \tag{3.1}$$

where $\boldsymbol{\mu}$ is a second order tensor and κ a scalar function, which depend on the plastic state and which describe kinematic and isotropic hardening. $\mathbf{E}'$ stands for $\mathbf{E} - \mathbf{E}''$, whereas $\mathbf{E}''$ is the plastic part of the strain tensor.

In general, the entire strain tensor or the rates of the strain tensor which arise from the kinematics of the given body motion, and the temperature and the temperature gradient, were considered as independent state variables. Since we have also introduced the plastic part of the strain tensor the second order tensor $\boldsymbol{\mu}$ and the scalar function κ as independent thermodynamic state variables, we need constitutive equations for these quantities.
tities.

In plasticity, it is not sufficient to assume equations only for $\mathbf{E}''$, $\boldsymbol{\mu}$ and κ. Rather, we have to introduce a condition which indicates that plastic deformations will occur. Moreover, additional conditions are necessary to guarantee plastic flow. All these conditions belong to the constitutive equations in plasticity. Mathematically, this is formulated in (3.2) to (3.18).

If the following condition (3.2)—called the yield or loading function—is satisfied, plastic deformations can take place:

$$F(\mathbf{S}, \boldsymbol{\mu}, \kappa, \theta) = 0. \tag{3.2}$$

If

$$F(\mathbf{S}, \boldsymbol{\mu}, \kappa, \theta) < 0 \tag{3.3}$$

no plastic deformations will occur.

To guarantee plastic flow during the thermodynamic process

$$\dot{F} = tr\left(\frac{\partial F}{\partial \mathbf{S}}\dot{\mathbf{S}}\right) + tr\left(\frac{\partial F}{\partial \boldsymbol{\mu}}\dot{\boldsymbol{\mu}}\right) + \frac{\partial F}{\partial \kappa}\dot{\kappa} + \frac{\partial F}{\partial \theta}\dot{\theta} = 0 \tag{3.4}$$

must be valid. Starting from (3.4), we derive certain loading criteria. When

$$F = 0 \text{ and } \tilde{F} > 0, \text{ then } \dot{\mathbf{E}}'' \neq 0,\ \dot{\boldsymbol{\mu}} \neq 0,\ \dot{\kappa} \neq 0,$$

when

$$F = 0 \text{ and } \tilde{F} < 0, \text{ then } \dot{\mathbf{E}}'' = 0,\ \dot{\boldsymbol{\mu}} = 0,\ \dot{\kappa} = 0, \tag{3.5}$$

when

$$F = 0 \text{ and } \tilde{F} = 0, \text{ then } \dot{\mathbf{E}}'' = 0,\ \dot{\boldsymbol{\mu}} = 0,\ \dot{\kappa} = 0.$$

where

$$\tilde{F} = tr\left(\frac{\partial F}{\partial \mathbf{S}}\dot{\mathbf{S}}\right) + \frac{\partial F}{\partial \theta}\dot{\theta}. \tag{3.6}$$

The requirement (3.5a) contains in connection with (3.4), that

$$-tr\left(\frac{\partial F}{\partial \boldsymbol{\mu}}\dot{\boldsymbol{\mu}}\right) - \frac{\partial F}{\partial \kappa}\dot{\kappa} > 0. \tag{3.7}$$

This inequality is important to derive an explicit form of the constitutive equation for the plastic strain rates, as we will soon see. Now, we postulate constitutive assumptions for the plastic part of the strain tensor and for the introduced quantities $\boldsymbol{\mu}$ and κ as follows:

$$\begin{aligned} \dot{\mathbf{E}}'' &= \bar{\dot{\mathbf{E}}}''(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g}, \dot{\mathbf{S}}, \dot{\theta}) \\ \dot{\boldsymbol{\mu}} &= \bar{\dot{\boldsymbol{\mu}}}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g}, \dot{\mathbf{S}}, \dot{\theta}) \\ \dot{\kappa} &= \bar{\dot{\kappa}}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g}, \dot{\mathbf{S}}, \dot{\theta}). \end{aligned} \tag{3.8}$$

Bearing in mind that classical plasticity theory is a time independent theory, we can specify (3.8) by

$$\begin{aligned} \dot{\mathbf{E}}'' &= \mathbf{A}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g})\dot{\mathbf{S}} + \mathbf{B}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g})\,\dot{\theta} \\ \dot{\boldsymbol{\mu}} &= \mathbf{I}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g})\dot{\mathbf{S}} + \mathbf{0}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g})\,\dot{\theta} \\ \dot{\kappa} &= tr[\mathbf{C}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g})\dot{\mathbf{S}}] + D(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta, \mathbf{g})\,\dot{\theta} \end{aligned} \tag{3.9}$$

where **A** and **I** are fourth-order tensors; **B**, **0** and **C** are second-order tensors, and D is a scalar.

The loading criteria (3.5) give some restrictions because of the assumptions (3.9). First let us regard the constitutive assumption (3.9a). From the loading requirements (3.5b) and (3.5c) we obtain, that $\dot{\mathbf{E}}''$ dissapears when $\tilde{F} \leq 0$. We replace therefore in (3.9a) **A** and **B** by

$$\mathbf{A} = \mathbf{M}\frac{\partial F}{\partial \mathbf{S}} \text{ and } \mathbf{B} = \mathbf{M}\frac{\partial F}{\partial \theta} \tag{3.10}$$

with $\mathbf{M}(\mathbf{E}', \mathbf{E}'', \boldsymbol{\mu}, \kappa, \theta)$ as a second order tensor. Then we derive from (3.9a) with (3.6)

$$\dot{\mathbf{E}}'' = \mathbf{M}\langle\tilde{F}\rangle. \tag{3.11}$$

The symbol $\langle\tilde{F}\rangle$ is defined as follows

$$\langle\tilde{F}\rangle = \begin{cases} 0 \text{ for } \tilde{F} \leq 0 \\ \tilde{F} \text{ for } \tilde{F} > 0. \end{cases} \tag{3.12}$$

Furthermore, the loading criteria (3.5a) and (3.5b) yield that $\dot{\boldsymbol{\mu}}$ is equal zero, if $\tilde{F} \leq 0$, or in view of (3.11), if $\dot{\mathbf{E}}''$ disappears. Therefore we set in (3.9b)

$$\mathbf{I} = c\,\mathbf{A} \text{ and } \mathbf{0} = c\,\mathbf{B}, \tag{3.13}$$

where c is a constant value, which describes material properties. Then we obtain from (3.9b)

$$\dot{\boldsymbol{\mu}} = c(\mathbf{A}\dot{\mathbf{S}} + \mathbf{B}\dot{\theta}) \tag{3.14}$$

or with (3.9a)

$$\dot{\boldsymbol{\mu}} = c\,\dot{\mathbf{E}}'' \tag{3.15}$$

which guarantees that $\dot{\mu}$ vanishes when $\dot{\mathbf{E}}''$ is equal to zero. The loading criteria (3.5b) and (3.5c) imply also that $\dot{\kappa}$ disappears if $\tilde{F} \leq 0$ or $\dot{\mathbf{E}}''$ is equal zero, respectively (eqn 3.11). A simple way of achieving this result from (3.9c) is to set in (3.9c)

$$\mathbf{C} = \mathbf{H}\,\mathbf{A}, \qquad D = tr(\mathbf{H}\,\mathbf{B})$$

with $\mathbf{H}(\mathbf{E}', \mathbf{E}'', \mu, \kappa, \theta)$ as a second-order tensor. Then we derive from (3.9c)

$$\dot{\kappa} = \mathbf{H}\,\mathbf{A}\,\dot{\mathbf{S}} + tr(\mathbf{H}\,\mathbf{B})\,\dot{\theta}$$

or with (3.9a)

$$\kappa = tr(\mathbf{H}\,\dot{\mathbf{E}}''). \tag{3.16}$$

In this form (3.16) $\dot{\kappa}$ disappears if $\dot{\mathbf{E}}''$ vanishes which is in accordance with the loading criteria (3.5)

Next, we are concerned with the determination of $\mathbf{M}$ in (3.11). Inequality (3.7) leads with (3.15), (3.16) and (3.11) to

$$-\langle \tilde{F} \rangle tr\left[\left(c\frac{\partial F}{\partial \mu} + \mathbf{H}\frac{\partial F}{\partial \kappa}\right)\mathbf{M}\right] > 0.$$

This inequality is satisfied if we choose

$$\mathbf{M} = -\frac{1}{h}\left(c\frac{\partial F}{\partial \mu} + \frac{\partial F}{\partial \kappa}\mathbf{H}\right). \tag{3.17}$$

where h is a positive scalar invariant, which can be determined by eqn (3.4). The final form of the constitutive equation for the plastic part of the strain tensor is then given by (3.15) and (3.17)

$$\dot{\mathbf{E}}'' = -\frac{1}{h}\left[tr\left(\frac{\partial F}{\partial \mathbf{S}}\dot{\mathbf{S}}\right) + \frac{\partial F}{\partial \theta}\dot{\theta}\right]\left(c\frac{\partial F}{\partial \mu} + \frac{\partial F}{\partial \kappa}\mathbf{H}\right). \tag{3.18}$$

If we choose F in such a way that

$$c\,\frac{\partial F}{\partial \mu} + \frac{\partial F}{\partial \kappa}\mathbf{H} = -\,a\frac{\partial F}{\partial \mathbf{S}}, \tag{3.19}$$

where a is a positive quantity, then we derive the form of the constitutive equation for $\dot{\mathbf{E}}''$ in the infinitesimal theory (see Naghdi[7])

$$\dot{\mathbf{E}}'' = +\frac{a}{h}\left[tr\left(\frac{\partial F}{\partial \mathbf{S}}\dot{\mathbf{S}}\right) + \frac{\partial F}{\partial \theta}\dot{\theta}\right]\frac{\partial F}{\partial \mathbf{S}}. \tag{3.20}$$

However, in (3.20) no limitation on the magnitude of the deformations has been introduced. The stress-strain relation (3.18) or (3.20) are the simplist forms that we can obtain if we consider linear relationships between the rates (see eqn (3.9)). For infinitesimal deformations, eqn (3.20) leads in the isothermal theory to the well known form of the plastic part of the strain tensor, first given by Melan[6] by considering the uniqueness theorem (see also Koiter[4]). Now we discuss restrictions that are imposed by the thermodynamical requirements. Following the usual procedure we obtain from (2.26) with (3.1)

$$\mathbf{S} = \rho\frac{\partial \bar{\psi}}{\partial \mathbf{E}'}, \qquad \eta = -\frac{\partial \bar{\psi}}{\partial \theta}, \qquad \frac{\partial \bar{\psi}}{\partial \mathbf{g}} = 0, \tag{3.21}$$

and the inequality

$$tr\left[\left(\mathbf{S} - \rho\frac{\partial \bar{\psi}}{\partial \mathbf{E}''} - \rho\frac{\partial \bar{\psi}}{\partial \mu}c - \rho\frac{\partial \bar{\psi}}{\partial \kappa}\mathbf{H}\right)\dot{\mathbf{E}}'' - \frac{1}{\theta}\mathbf{q}\mathbf{g} \geq 0, \tag{3.22}$$

if we consider (3.15) and (3.16). With the results (3.21), the balance of energy (2.25) reduces to

$$\rho\dot{r} - \rho\dot{\eta}\theta - \operatorname{div}\dot{\mathbf{q}} + tr\left[\left(\mathbf{S} - \rho\frac{\partial\bar{\psi}}{\partial\mathbf{E}''} - \rho\frac{\partial\bar{\psi}}{\partial\boldsymbol{\mu}}c - \rho\frac{\partial\bar{\psi}}{\partial\kappa}\mathbf{H}\right)\dot{\mathbf{E}}''\right] = 0. \tag{3.23}$$

In the special case, if $\bar{\Psi}$ is independent of $\mathbf{E}''$, $\boldsymbol{\mu}$ and κ, (3.22) and (3.23) simplify to

$$tr(\mathbf{S}\dot{\mathbf{E}}'') - \frac{1}{\theta}\dot{\mathbf{q}}\mathbf{g} \geq 0, \tag{3.24}$$

$$\rho\dot{r} - \rho\dot{\eta}\theta - \operatorname{div}\dot{\mathbf{q}} + tr(\mathbf{S}\cdot\dot{\mathbf{E}}'') = 0. \tag{3.25}$$

Inequality (3.22) or (3.24) give restrictions on the loading function F. If we substitute $\dot{\mathbf{E}}''$ by (3.18) we obtain from (3.24)

$$-\frac{1}{h}\left[tr\left(\frac{\partial F}{\partial\mathbf{S}}\dot{\mathbf{S}}\right) + \frac{\partial F}{\partial\theta}\dot{\theta}\right] tr\left[\left(c\frac{\partial F}{\partial\boldsymbol{\mu}} + \frac{\partial F}{\partial\kappa}\mathbf{H}\right)\mathbf{S}\right] - \frac{1}{\theta}\dot{\mathbf{q}}\mathbf{g} \geq 0$$

or for $\dot{\mathbf{q}}$ equal zero, omitting the positive quantity h and the positive value in the first bracket,

$$-tr\left[\left(c\frac{\partial F}{\partial\boldsymbol{\mu}} + \frac{\partial F}{\partial\kappa}\mathbf{H}\right)\mathbf{S}\right] \geq 0, \tag{3.26}$$

which clearly shows, that F must be chosen in such a way that inequality (3.26) is fulfilled.

4. ELASTO-PLASTIC MATERIAL WITH VISCOUS PROPERTIES

4.1 *Introduction*

Before developing the constitutive equations and restrictions, it seems necessary to analyze the viscoplastic behavior of the materials from the physical point of view. We do this because the viscous behavior in the plastic region is complex and the possibilities to combine the viscous with the elastic and plastic properties are many. We consider first the behavior of the material under dynamic loading. Then, the beginning of plastic yield will be delayed. The delay depends on the strain rate. Therefore, plastic behavior is influenced by the rate of strain and hence a viscous effect result. This behavior has been studied in recent years in several papers [10, 11], particularly in view of general theorems [1], and has led to the theory of rate sensitive materials.

Steel, at high temperature, and other materials such as clay show other viscous properties. Beside elastic and plastic, viscous effects appear. The entire strain can be decomposed into an initial strain which contains the elastic and plastic properties, and into the viscous strain [9]. The time history of the strain can be decomposed into a primary, secondary and tertiary region [9]. The secondary region is of special interest because the strain depends linearly on time. The description of the behavior of the material in the secondary region remains difficult because the rate of the strains depends non-linearly on the stresses. The following thermodynamic considerations are restricted to such viscoplastic materials described above.

4.2 *Elastic-viscoplastic material*

A material may be defined as an elastic-viscoplastic material, when it is described by a set of constitutive equations in the following sense:

$$\psi = \hat{\psi}(\mathbf{E}', \mathbf{E}''^{v}, \alpha, \theta, \mathbf{g}),\dagger$$

$$\eta = \hat{\eta}(\mathbf{E}', \mathbf{E}''^{v}, \alpha, \theta, \mathbf{g}),$$

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{E}', \mathbf{E}''^{v}, \alpha, \theta, \mathbf{g}),$$

$$\dot{\mathbf{q}} = \hat{\dot{\mathbf{q}}}(\mathbf{E}', \mathbf{E}''^{v}, \alpha, \theta, \mathbf{g}), \tag{4.1}$$

†We can also introduce an second order tensor which describes anisotropic effects, as in inviscid plasticity, but for simplicity we suppres this.

where $\mathbf{E}''^{v}$ denotes the viscoplastic part of the strain tensor and α a scalar value function. Both values, $\mathbf{E}''^{v}$ and α, must be determined by constitutive equations, which will be formulated in the next considerations. $\mathbf{E}'$ stands for $\mathbf{E}-\mathbf{E}''^{v}$.

As in inviscid plasticity we have to introduce a condition which indicates when viscoplastic deformations can take place. Viscoplastic deformations can occur, if the following condition is fulfilled

$$L(\mathbf{S},\alpha,\theta)=G(\dot{\mathbf{E}}''^{v},\dot{\alpha}), \tag{4.2}$$

where G is a positive function of $\dot{\mathbf{E}}''^{v}$ and $\dot{\alpha}$ which is equal to zero when $\dot{\mathbf{E}}''^{v}$ reaches zero. Equation (4.2) is in accordance with the observation of the physical behavior, described in Section 4.1, that the beginning of viscoplastic flow is delayed and the delay depends on the strain rates. To guarantee viscoplastic flow the condition

$$tr\left(\frac{\partial L}{\partial \mathbf{S}}\dot{\mathbf{S}}\right)+\frac{\partial L}{\partial \theta}\dot{\theta}=tr\left(\frac{\partial G}{\partial \dot{\mathbf{E}}''^{v}}\ddot{\mathbf{E}}''^{v}\right)+\frac{\partial G}{\partial \dot{\alpha}}\ddot{\alpha}-\frac{\partial L}{\partial \alpha}\dot{\alpha} \tag{4.3}$$

must be valid. Following the considerations in inviscid plasticity (Section 3) we cannot derive restrictions on the nature of the constitutive equation for the viscoplastic part of the strain tensor, because as known from experimental results, the loading conditions

$$tr\left(\frac{\partial L}{\partial \mathbf{S}}\dot{\mathbf{S}}\right)+\frac{\partial L}{\partial \theta}\dot{\theta}\gtreqless 0 \tag{4.4}$$

always lead to viscoplastic deformations. However, in the limit when $\dot{E}''^{v}$ disappears, the following loading criteria must be valid:

$$\begin{aligned} &L=0 \text{ and } \tilde{L}>0, \text{ then } \ddot{\mathbf{E}}''^{v}\neq 0, \quad \ddot{\alpha}\neq 0,\\ &L=0 \text{ and } \tilde{L}<0, \text{ then } \ddot{\mathbf{E}}''^{v}=0, \quad \ddot{\alpha}=0,\\ &L=0 \text{ and } \tilde{L}=0, \text{ then } \ddot{\mathbf{E}}''^{v}=0, \quad \ddot{\alpha}=0, \end{aligned} \tag{4.5}$$

where

$$\tilde{L}=tr\left(\frac{\partial L}{\partial \mathbf{S}}\dot{\mathbf{S}}\right)+\frac{\partial L}{\partial \theta}\dot{\theta}. \tag{4.6}$$

Now, we introduce the constitutive assumptions for $\mathbf{E}''^{v}$ and α.

$$\begin{aligned} \dot{\mathbf{E}}''^{v}&=\hat{\dot{\mathbf{E}}}''^{v}(\mathbf{E}',\mathbf{E}''^{v},\alpha,\theta,\mathbf{g}),\\ \ddot{\mathbf{E}}''^{v}&=\hat{\ddot{\mathbf{E}}}''^{v}(\mathbf{E}',\mathbf{E}''^{v},\alpha,\theta,\mathbf{g},\dot{\mathbf{S}},\dot{\theta}) \end{aligned} \tag{4.7}$$

where we specify (4.7b) by

$$\ddot{\mathbf{E}}''^{v}=\mathbf{R}(\mathbf{E}',\mathbf{E}''^{v},\alpha,\theta,\mathbf{g})\dot{\mathbf{S}}+\mathbf{T}(\mathbf{E}',\mathbf{E}''^{v},\alpha,\theta,\mathbf{g})\,\dot{\theta}, \tag{4.8}$$

with $\mathbf{R}$ and $\mathbf{T}$ as fourth- and second-order tensors.

Furthermore

$$\begin{aligned} \dot{\alpha}&=\hat{\dot{\alpha}}(\mathbf{E}',\mathbf{E}''^{v},\alpha,\theta,\mathbf{g}),\\ \ddot{\alpha}&=\hat{\ddot{\alpha}}(\mathbf{E}',\mathbf{E}''^{v},\alpha,\theta,\mathbf{g},\dot{\mathbf{S}},\dot{\theta}). \end{aligned} \tag{4.9}$$

The introduction of the constitutive equations for the viscoplastic strain is necessary in the above sense, because the constitutive equations for the description of the viscous properties must be time-dependent.

Similar to the considerations in Section 3 we set

$$\dot{\alpha} = tr(\mathbf{W}\dot{\mathbf{E}}''^{v}), \qquad \ddot{\alpha} = tr(\mathbf{V}\ddot{\mathbf{E}}''^{v}), \tag{4.10}$$

where $\mathbf{W}$ and $\mathbf{V}$ are second-order tensors, which depend on $\mathbf{E}'$, $\mathbf{E}''^{v}$, α, θ, $\mathbf{g}$. The loading condition (4.5a) requires in connection with (4.10) that for $\dot{\mathbf{E}}''^{v} = 0$

$$\left(\frac{\partial G}{\partial \dot{\mathbf{E}}''^{v}} + \frac{\partial G}{\partial \dot{\alpha}}\mathbf{V}\right)\ddot{\mathbf{E}}''^{v} > 0. \tag{4.11}$$

This inequality (4.11) will be used to derive an explicit form for $\ddot{\mathbf{E}}''^{v}$. First however, we simplify eqn (4.8). The constitutive eqn (4.8) has to describe the loading condition (4.5c). Therefore we set

$$\mathbf{R} = \mathbf{N}\frac{\partial L}{\partial \mathbf{S}}, \qquad \mathbf{T} = \mathbf{N}\frac{\partial L}{\partial \theta}, \tag{4.12}$$

where $\mathbf{N}$ is a second-order tensor. With (4.12) we obtain from (4.8)

$$\ddot{\mathbf{E}}''^{v} = \mathbf{N}\left[tr\left(\frac{\partial L}{\partial S}\dot{\mathbf{S}}\right) + \frac{\partial L}{\partial \theta}\dot{\theta}\right]. \tag{4.13}$$

The second-order tensor $\mathbf{N}$ will be determined in such a way that inequality (4.11) is always valid. This is surely given, if

$$\mathbf{N} = \frac{1}{m}\left(\frac{\partial G}{\partial \dot{\mathbf{E}}''^{v}} + \frac{\partial G}{\partial \dot{\alpha}}\mathbf{V}\right), \tag{4.14}$$

with m as a positive scalar value. Equation (4.13) yield with (4.14) the final form of the constitutive equation for $\ddot{\mathbf{E}}''^{v}$:

$$\ddot{\mathbf{E}}''^{v} = \frac{1}{m}\left[tr\left(\frac{\partial L}{\partial S}\dot{\mathbf{S}}\right) + \frac{\partial L}{\partial \theta}\dot{\theta}\right]\left(\frac{\partial G}{\partial \dot{\mathbf{E}}''^{v}} + \frac{\partial G}{\partial \dot{\alpha}}\mathbf{V}\right). \tag{4.15}$$

If we require that

$$\frac{\partial G}{\partial \dot{\mathbf{E}}''^{v}} + \frac{\partial G}{\partial \dot{\alpha}}\mathbf{V} = b\frac{\partial L}{\partial \mathbf{S}}, \tag{4.16}$$

with b as a constant value, then the constitutive equation can be replaced by

$$\ddot{\mathbf{E}}''^{v} = \frac{b}{m}\left[tr\left(\frac{\partial L}{\partial \mathbf{S}}\dot{\mathbf{S}}\right) + \frac{\partial L}{\partial \theta}\dot{\theta}\right]\frac{\partial L}{\partial \mathbf{S}}, \tag{4.17}$$

as pointed out in Section 3.

If we extend (4.17) in such a manner that in accordance with the constitutive assumptions (4.8) and the above requirements

$$\ddot{\mathbf{E}}''^{v} = \frac{\beta}{2}\frac{\partial\phi(L)}{\partial L}\left[tr\left(\frac{\partial L}{\partial \mathbf{S}}\dot{\mathbf{S}}\right) + \frac{\partial L}{\partial \theta}\dot{\theta}\right]\frac{\partial L}{\partial \mathbf{S}} + \frac{\beta}{2}\phi(L)\left[\frac{\partial^2 L}{\partial \mathbf{S}\partial \mathbf{S}}\dot{S} + \frac{\partial^2 L}{\partial \mathbf{S}\partial \theta}\dot{\theta}\right], \tag{4.18}$$

then by integration, eqn (4.18) can be written as

$$\dot{\mathbf{E}}''^{v} = \frac{\beta}{2}\phi(L)\frac{\partial L}{\partial \mathbf{S}}, \tag{4.19}$$

which has the form assumed by Perzyna[10]. In this equation β is a material dependent number, and $\phi(L)$ a nonlinear function. Comparing (4.18) and (4.17) the nonlinear function $\phi(L)$ must vanish for $L=0$ and must have a positive slope. The postulate $(\partial\phi/\partial L)>0$ is also necessary to fulfill general theorems in viscoplasticity ([1]).

If the elastic-viscoplastic material loses its viscous properties, then the viscoplastic strain rate (4.19) must be equal to the plastic strain rate, when $L = F = 0$. Perzyna concludes in [11] that in this limit the coefficient $\frac{1}{2}\beta\phi(L)$ has to yield an undetermined value.

From the C-D inequality we can derive restrictions on the loading function. Starting with (4.1) we obtain from the C–D inequality in case a (2.26)

$$\mathbf{S}=\rho\frac{\partial\hat{\psi}}{\partial\mathbf{E}'},\qquad \eta=-\frac{\partial\hat{\psi}}{\partial\theta},\qquad \frac{\partial\hat{\psi}}{\partial\mathbf{g}}=0, \tag{4.20}$$

and the inequality

$$tr[(\mathbf{S}-\rho\hat{\psi}_{,\mathbf{E}''^{v}}-\rho\hat{\psi}_{,\alpha}\mathbf{W})\dot{\mathbf{E}}''^{v}]-\frac{1}{\theta}\dot{\mathbf{q}}\mathbf{g}\geq 0, \tag{4.21}$$

whereas we have used eqn (4.10). The balance of energy simplifies to

$$\rho\dot{r}-\rho\dot{\eta}\theta-\operatorname{div}\dot{\mathbf{q}}+tr\left[\left(\mathbf{S}-\rho\frac{\partial\hat{\psi}}{\partial\mathbf{E}''^{v}}-\partial\frac{\partial\hat{\psi}}{\partial\alpha}\mathbf{W}\right)\dot{\mathbf{E}}''^{v}\right]=0. \tag{4.22}$$

Since in the limit $\dot{\mathbf{E}}''^{v}$ equals zero the expressions in the brackets vanish, no restrictions upon the viscoplastic deformations can be obtained in case *a*. Therefore, we proceed to case *b*. With the results (4.20) the C–D-inequality in the form (2.27) yield after some calculations

$$\begin{aligned}&\frac{\dot{\theta}}{\theta}(\rho\dot{r}-\rho\dot{\eta}\theta-\operatorname{div}\dot{\mathbf{q}})+tr\left[\left(\dot{\mathbf{S}}-\rho\frac{\partial\dot{\psi}}{\partial\mathbf{E}''^{v}}-\rho\frac{\partial\dot{\psi}}{\partial\alpha}\mathbf{W}\right)\dot{\mathbf{E}}''^{v}\right]\\&\qquad+tr\left[\left(\mathbf{S}-\rho\frac{\partial\hat{\psi}}{\partial\mathbf{E}''^{v}}-\rho\frac{\partial\hat{\psi}}{\partial\alpha}\mathbf{V}\right)\ddot{\mathbf{E}}''^{v}\right]-\theta\left[\frac{1}{\theta^{2}}\dot{\mathbf{q}}\mathbf{g}\right]^{\cdot}\geq 0,\end{aligned} \tag{4.23}$$

and with (4.22)

$$\begin{aligned}&-\frac{\dot{\theta}}{\theta}tr[(\mathbf{S}-\rho\hat{\psi}_{,\mathbf{E}''^{v}}-\rho\hat{\psi}_{,\alpha}\mathbf{W})\dot{\mathbf{E}}''^{v}]+tr[(\dot{\mathbf{S}}-\rho\dot{\psi}_{,\mathbf{E}''^{v}}-\rho\dot{\psi}_{,\alpha}\mathbf{W})\dot{\mathbf{E}}''^{v}]\\&+tr[(\mathbf{S}-\rho\hat{\psi}_{,\mathbf{E}''^{v}}-\rho\hat{\psi}_{,\alpha}\mathbf{V})\ddot{\mathbf{E}}''^{v}]-\theta\left[\frac{1}{\theta^{2}}\dot{\mathbf{q}}\,\mathbf{g}\right]^{\cdot}\geq 0.\end{aligned} \tag{4.24}$$

Taking the limit as $\dot{\mathbf{E}}''^{v}\to$ zero, inequality (4.24) reduces to

$$tr\left[\left(\mathbf{S}-\rho\frac{\partial\hat{\psi}}{\partial\mathbf{E}''^{v}}-\rho\frac{\partial\hat{\psi}}{\partial\alpha}\mathbf{V}\right)\ddot{\mathbf{E}}''^{v}-\theta\left[\frac{1}{\theta^{2}}\dot{\mathbf{q}}\mathbf{g}\right]^{\cdot}\geq 0. \tag{4.25}$$

If we assume that $\hat{\Psi}$ is independent of $\mathbf{E}''^{v}$ and α, and $\dot{\mathbf{q}}$ and the time derivations of $\dot{\mathbf{q}}$ vanish, then we obtain with (4.15)

$$\frac{1}{m}\left[tr\left(\frac{\partial L}{\partial\mathbf{S}}\dot{\mathbf{S}}\right)+\frac{\partial L}{\partial\theta}\dot{\theta}\right]tr\left[\left(\frac{\partial G}{\partial\mathbf{E}''^{v}}+\frac{\partial G}{\partial\dot{\alpha}}\mathbf{V}\right)\mathbf{S}\right]\geq 0. \tag{4.26}$$

Omitting the positive values m and in the first bracket, inequality (4.26) leads to

$$tr\left[\left(\frac{\partial G}{\partial\mathbf{E}''^{v}}+\frac{\partial G}{\partial\dot{\alpha}}\mathbf{V}\right)\mathbf{S}\right]\geq 0, \tag{4.27}$$

which contains a restriction on the function G.

4.3 ELASTIC-PLASTIC-VISCOUS MATERIAL

An elastic-plastic-viscous material may be characterized by the following set of constitutive equations.

$$\begin{aligned}\psi &= \bar{\psi}(\mathbf{E}', \mathbf{E}'', \mathbf{E}^v, \kappa, \varphi, \theta, \mathbf{g}),\dagger \\ \eta &= \bar{\eta}(\mathbf{E}', \mathbf{E}'', \mathbf{E}^v, \kappa, \varphi, \theta, \mathbf{g}), \\ \mathbf{S} &= \bar{\mathbf{S}}(\mathbf{E}', \mathbf{E}'', \mathbf{E}^v, \kappa, \varphi, \theta, \mathbf{g}), \\ \dot{\mathbf{q}} &= \bar{\mathbf{q}}(\mathbf{E}', \mathbf{E}'', \mathbf{E}^v, \kappa, \varphi, \theta, \mathbf{g}). \end{aligned} \quad (4.28)$$

$\mathbf{E}^v$ stands for the viscous part of the strain tensor and φ is a scalar value function. $\mathbf{E}'$ is equal $\mathbf{E}-\mathbf{E}''-\mathbf{E}^v$. Viscous deformations can take place, if the following condition is satisfied

$$f(\mathbf{S}, \kappa_0, \varphi, \theta) = P(\dot{\mathbf{E}}^v, \dot{\varphi}), \quad (4.29)$$

where, in general, κ_0 is prescribed by the constitutive eqns (3.16) and (3.13) respectively. However, for several materials as steel under high temperature, the quantity κ_0 can be considered as a constant. We postulate, that P is always positive and vanishes if $\dot{\mathbf{E}}^v$ and $\dot{\varphi}$ are equal zero. Besides, the following condition must also be valid:

$$tr\left(\frac{\partial f}{\partial \mathbf{S}}\dot{\mathbf{S}}\right) + \frac{\partial f}{\partial \theta}\dot{\theta} = \frac{\partial P}{\partial \dot{\mathbf{E}}^v}\ddot{\mathbf{E}}^v + \frac{\partial P}{\partial \dot{\varphi}}\ddot{\varphi} - \frac{\partial f}{\partial \varphi}\dot{\varphi}, \quad (4.30)$$

when κ_0 is a constant. The loading conditions

$$\tilde{f} = tr\left(\frac{\partial f}{\partial S}\dot{\mathbf{S}}\right) + \frac{\partial f}{\partial \theta}\dot{\theta} \gtreqless 0 \quad (4.31)$$

lead always to viscous deformations, if (4.29) is satisfied, as experimental results show. Therefore we proceed to the state where $\dot{\mathbf{E}}^v$ disappears. Hence

$$\begin{aligned} &f = 0 \text{ and } \tilde{f} > 0, \text{ then } \ddot{\mathbf{E}}^v \neq 0, \quad \ddot{\varphi} \neq 0, \\ &f = 0 \text{ and } \tilde{f} < 0, \text{ then } \ddot{\mathbf{E}}^v = 0, \quad \ddot{\varphi} = 0, \\ &f = 0 \text{ and } \tilde{f} = 0, \text{ then } \ddot{\mathbf{E}}^v = 0; \quad \ddot{\varphi} = 0. \end{aligned} \quad (4.32)$$

In the following considerations we suppress the determination of the plastic part of the deformations because this has been done in Section 3. For the viscous part of the deformations we introduce the following constitutive equations:

$$\begin{aligned} \dot{\mathbf{E}}^v &= \bar{\dot{\mathbf{E}}}^v(\mathbf{E}', \mathbf{E}', \mathbf{E}^v, \kappa, \varphi, \theta, \mathbf{g}), \\ \ddot{\mathbf{E}}^v &= \bar{\ddot{\mathbf{E}}}^v(\mathbf{E}', \mathbf{E}'', \mathbf{E}^v, \kappa, \varphi, \theta, \mathbf{g}, \dot{\mathbf{S}}, \dot{\theta}). \end{aligned} \quad (4.33)$$

Equation (4.33b) may be specializied by

$$\ddot{\mathbf{E}}^v = \mathbf{K}\,\dot{\mathbf{S}} + \mathbf{Q}\,\dot{\theta}, \quad (4.34)$$

with $\mathbf{K}$ and $\mathbf{Q}$ as fourth- and second-order tensors which depend on $\mathbf{E}'$, $\mathbf{E}''$, $\mathbf{E}^v$, κ, φ, θ, $\mathbf{g}$. As pointed out in Section 3 we set

$$\dot{\varphi} = tr(\mathbf{y}\dot{\mathbf{E}}^v), \qquad \ddot{\varphi} = tr(\mathbf{Z}\,\ddot{\mathbf{E}}^v). \quad (4.35)$$

The second-order tensors $\mathbf{y}$ and $\mathbf{Z}$ depend on $\mathbf{E}'$, $\mathbf{E}''$, $\mathbf{E}^v$, κ, φ, θ, $\mathbf{g}$, the condition (4.30) leads

†As in Section 3 we can also include a second order tensor which describes anisotropic properties.

with the loading requirement (4.32a) to the following inequality for the state $\dot{\mathbf{E}}^v$ equal zero:

$$\frac{\partial P}{\partial \dot{\mathbf{E}}^v}\ddot{\mathbf{E}}^v + \frac{\partial P}{\partial \dot{\varphi}}\ddot{\varphi} > 0. \tag{4.36}$$

Inequality (4.36) leads with eqn (4.35b) to

$$\left(\frac{\partial P}{\partial \dot{\mathbf{E}}^v} + \frac{\partial P}{\partial \dot{\varphi}}\mathbf{Z}\right)\ddot{\mathbf{E}}^v > 0. \tag{4.37}$$

Now, we proceed to give an explicit form for $\ddot{\mathbf{E}}^v$. Following the previously discussed developments in Section 3 and 4.2 we obtain for $\ddot{\mathbf{E}}^v$:

$$\ddot{\mathbf{E}}^v = \mathbf{U}\left[tr\left(\frac{\partial f}{\partial \mathbf{S}}\dot{\mathbf{S}}\right) + \frac{\partial f}{\partial \theta}\dot{\theta}\right], \tag{4.38}$$

where $\mathbf{U}$ is a second-order tensor, depending on $\mathbf{E}'$, $\mathbf{E}''$, $\mathbf{E}^v$, κ, φ, θ, $\mathbf{g}$. With this equation we can describe the loading state (4.32b, c). To fulfill inequality (4.37) we set

$$\ddot{\mathbf{E}}^v = \frac{1}{n}\left[tr\left(\frac{\partial f}{\partial \mathbf{S}}\dot{\mathbf{S}}\right) + \frac{\partial f}{\partial \theta}\dot{\theta}\right]\left(\frac{\partial P}{\partial \dot{\mathbf{E}}^v} + \frac{\partial P}{\partial \dot{\varphi}}\mathbf{Z}\right), \tag{4.39}$$

with n as a positive scalar value. This final form for $\ddot{\mathbf{E}}^v$ can be replaced by the following equation, if we assume that

$$\frac{\partial P}{\partial \dot{\mathbf{E}}^v} + \frac{\partial P}{\partial \dot{\varphi}}\mathbf{Z} = \mathrm{d}\frac{\partial f}{\partial \mathbf{S}}, \tag{4.40}$$

where d is a constant value. Then we obtain from (4.39) with (4.40)

$$\ddot{\mathbf{E}}^v = \frac{\mathrm{d}}{n}\left[fr\left(\frac{\partial f}{\partial \mathbf{S}}\dot{\mathbf{S}}\right) + \frac{\partial f}{\partial \theta}\dot{\theta}\right]\frac{\partial f}{\partial \mathbf{S}}. \tag{4.41}$$

Above all, if we follow the development in Section 4.2, we can derive a constitutive equation for $\dot{\mathbf{E}}^v$:

$$\dot{\mathbf{E}}^v = \frac{\gamma}{2}\psi(f)\frac{\partial f}{\partial \mathbf{S}}. \tag{4.42}$$

In this equation γ is a material dependent number and $\psi(f)$ a nonlinear function, which vanishes, if f is equal zero, and which has a positive slope. The constitutive eqn (4.42) contains Nortons law (see [9]). Next, we are concerned with the derivation of restrictions on the viscous deformations, based on the C–D inequality. In case a the C–D inequality (2.26) yield with (4.28)

$$\mathbf{S} = \rho\frac{\partial \bar{\psi}}{\partial \mathbf{E}'}, \qquad \eta = -\frac{\partial \bar{\psi}}{\partial \theta}, \qquad \frac{\partial \bar{\psi}}{\partial \mathbf{g}} = 0, \tag{4.43}$$

and the inequality

$$tr\left[\left(\mathbf{S} - \rho\frac{\partial \bar{\psi}}{\partial \mathbf{E}''} - \rho\frac{\partial \bar{\psi}}{\partial \kappa}\mathbf{H}\right)\dot{\mathbf{E}}''\right] + tr\left[\left(\mathbf{S} - \rho\frac{\partial \bar{\psi}}{\partial \mathbf{E}^v} - \rho\frac{\partial \bar{\psi}}{\partial \varphi}\mathbf{y}\right)\dot{\mathbf{E}}^v\right] - \frac{1}{\theta}\dot{\mathbf{q}}\,\mathbf{g} \geq 0. \tag{4.44}$$

The balance of energy reduces in case a to

$$\rho\dot{r} - \rho\dot{\eta}\theta - \operatorname{div}\dot{\mathbf{q}} + tr\left[\left(\mathbf{S} - \rho\frac{\partial \bar{\psi}}{\partial \mathbf{E}''} - \rho\frac{\partial \bar{\psi}}{\partial \kappa}\mathbf{H}\right)\dot{\mathbf{E}}''\right] + tr\left[\left(\mathbf{S} - \rho\frac{\partial \bar{\psi}}{\partial \mathbf{E}^v} - \rho\frac{\partial \bar{\psi}}{\partial \varphi}\mathbf{y}\right)\dot{\mathbf{E}}^v\right] = 0. \tag{4.45}$$

For the previously discussed state $\dot{\mathbf{E}}^v$ equal to zero, inequality (4.44) gives no restrictions upon the viscous deformations. Therefore, we regard case b in Section 2.

Considering the results (4.43) we obtain after some calculations from the C–D inequality in the form (2.27), if we use (4.45) and regard the state $\dot{\mathbf{E}}^v$ equal to zero

$$-\frac{\dot{\theta}}{\theta}\,tr\Big[\Big(\mathbf{S}-\rho\frac{\partial\bar{\psi}}{\partial\mathbf{E}''}-\rho\frac{\partial\bar{\psi}}{\partial\kappa}\mathbf{H}\Big)\dot{\mathbf{E}}''+\Big\{tr\Big[\Big(\mathbf{S}-\rho\frac{\partial\bar{\psi}}{\partial\mathbf{E}''}-\rho\frac{\partial\bar{\psi}}{\partial\kappa}\mathbf{H}\Big)\dot{\mathbf{E}}''\Big\}^{\cdot}$$

$$+\,tr\Big[\Big(\mathbf{S}-\rho\frac{\partial\bar{\psi}}{\partial E^v}-\rho\frac{\partial\bar{\psi}}{\partial\varphi}\mathbf{Z}\Big)\ddot{\mathbf{E}}^v\Big]-\theta\Big[\frac{1}{\theta^2}\dot{\mathbf{q}}\mathbf{g}\Big]^{\cdot}\geq 0. \qquad (4.46)$$

We assume that the sum of the first two terms on the left side of inequality (4.46) is always positive. Then inequality (4.46) is satisfied, if

$$tr\Big[\Big(\mathbf{S}-\rho\frac{\partial\bar{\psi}}{\partial\mathbf{E}^v}-\rho\frac{\partial\bar{\psi}}{\partial\varphi}\mathbf{Z}\Big)\ddot{\mathbf{E}}^v\Big]-\theta\Big[\frac{1}{\theta^2}\dot{\mathbf{q}}\mathbf{g}\Big]^{\cdot}\geq 0. \qquad (4.47)$$

Recalling the result (4.39), inequality (4.47) gives restrictions in view of the function P.

5. FINAL REMARKS

Within the framework of non-isothermal and finite deformations, basic constitutive equations of plastic materials without and with viscous properties are developed in this paper. This seems to be necessary, because the inviscid plasticity theory and the viscoplasticity theory contain some open questions with regard to the formulation of the constitutive equations: e.g. the plastic strain rates depend on a second-order tensor which is unknown (see [3], [11]). Since the restrictions in plasticity theory without and with viscous properties, which are needed to obtain consistent constitutive equations, are only valid for infinitesimal or isothermal deformations (see [1, 4, 6, 8]), it is not possible to give an explicit form of the plastic part of the strain tensor. Therefore, in this paper a basic inequality is derived from the loading criteria. With the help of this inequality we can give an explicit form of the constitutive equation for the plastic strain rates which is similar to the concept of plastic potential in the infinitesimal inviscid plasticity theory. In viscoplasticity theory we can derive, in some special cases, a similar expression for the second time derivative of the inelastic part of the strain tensor.

The basic constitutive equations of the inviscid plasticity theory contain the results of the infinitestimal theory [12] and the constitutive equations in viscoplasticity theory the constitutive equation for "rate-sensitive" materials (see [10]) and Norton's law (see [9]).

It is well known that the Clausius-Duhem inequality gives some restrictions in view of the constitutive equations. For our purpose we have to expand all quantities in the balance of energy and in the C–D inequality in series. This procedure leads in our development to restrictions only on the loading functions. The use of the Taylor series expansion seems to be new.

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